The Gaussian Curvature Map

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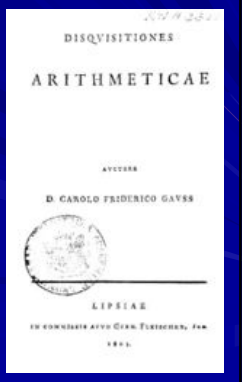
**Author is a full time employee of Optikon 2000 Spa (Int.: C3)*

Historical background:

- Theorema Egregium by Carl Friderich Gauss


DISQVISITIONES
ARITHMETICAE
AVCTORE
D. CAROLO FRIDERICO GAVSS
LIPSIAE

Historical background:

- C.F. Gauss (1777-1855) was professor of mathematics in Göttingen (D). Has developed theories adopted in
  - Analysis
  - Number theory
  - Differential Geometry
  - Statistic (Guaussian distribution)
  - Physics (magnetism, electrostatics, optics, Astronomy)

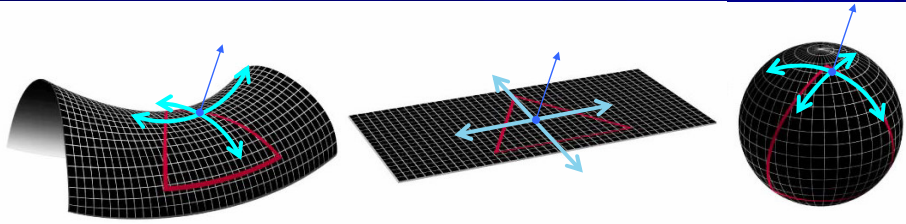
"*Almost everything, which the mathematics of our century has brought forth in the way of original scientific ideas, attaches to the name of Gauss.*"

Kronecker, L.

Gaussian Curvature of a surface

(Definition)

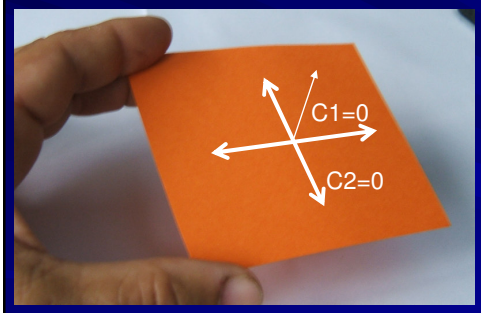
- Gaussian curvature is the product of steepest and flattest local curvatures, at each point.

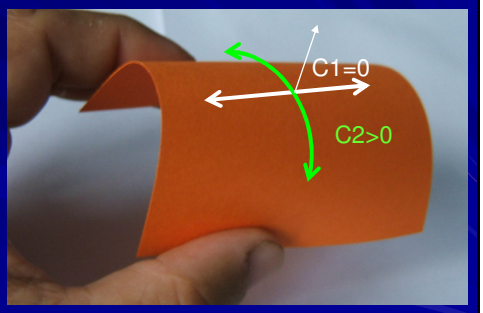


Negative GC Zero GC Positive GC

Gauss' *Theorema Egregium* (1828)

- Gaussian curvature (C1 x C2) of a flexible surface is invariant. (non-elastic isometric distortion)



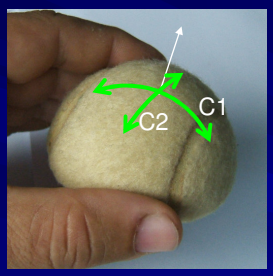


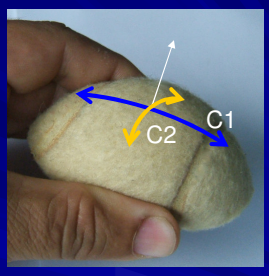
C1 x C2 = 0

C1 x C2 = 0

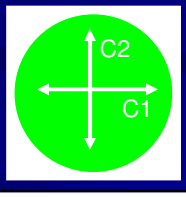
Gauss' *Theorema Egregium*

- Gaussian curvature is an intrinsic property of surfaces

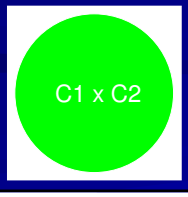




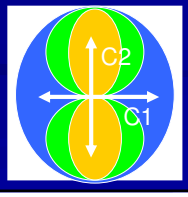
Curvature



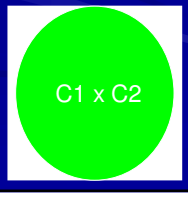
Gaussian



Curvature



Gaussian



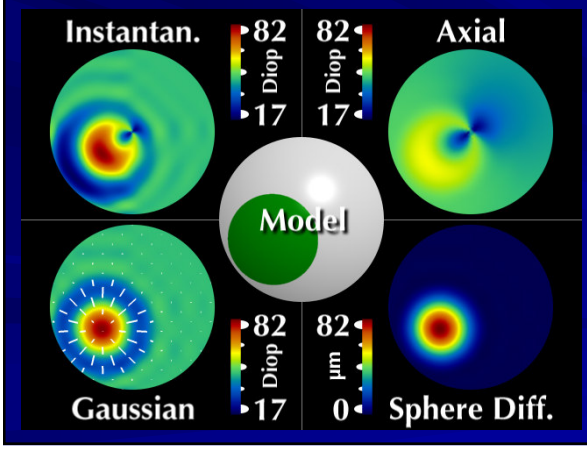
Gauss' *Theorema Egregium*

- "*A consequence of the Theorema Egregium is that the Earth cannot be displayed on a map without distortion. The Mercator projection… preserves angles but fails to preserve area.*" (Wikipedia)

The Gaussian Map in Corneal Topography

Proposed by B.Barsky et al. in 1997

- "Gaussian power with cylinder vector field representation for corneal topography maps". *Optom.Vis.Sci. 74-1997*



Other proposals

- T.Turner (Orbscan, 1995) – Mean curvature maps
- M.Tang et alt (AJO Dec. 2005) – Mean curvature maps
- R.Mattioli (Keratron, Refractive-on-line 2007) - Gaussian maps

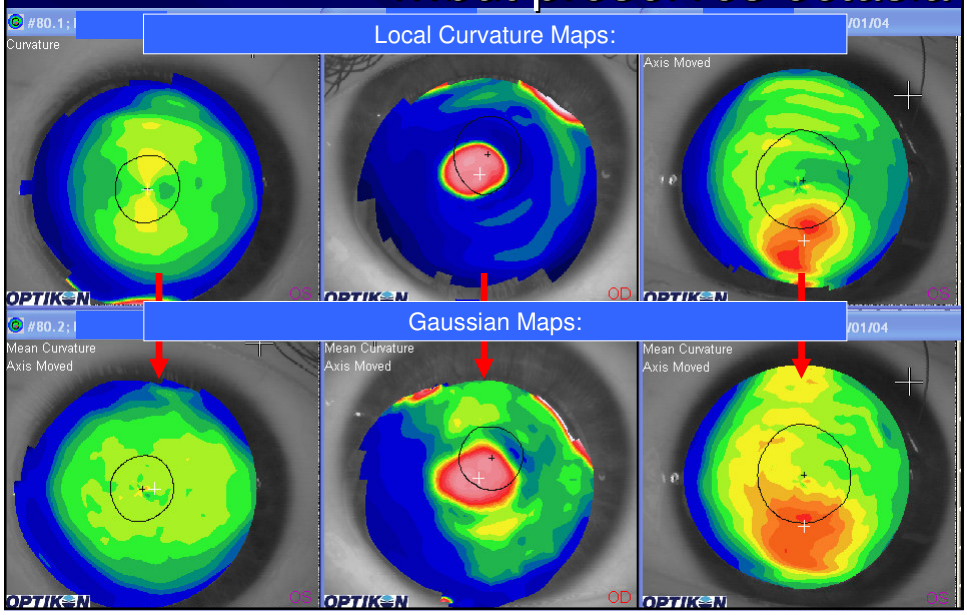
Mean is local curvature arithmetic average (D).
Gaussian is curvature geometric average (D):

$$\text{Mean} = (C1+C2)/2 \qquad \text{Gaussian} = \sqrt{C1 \times C2}$$

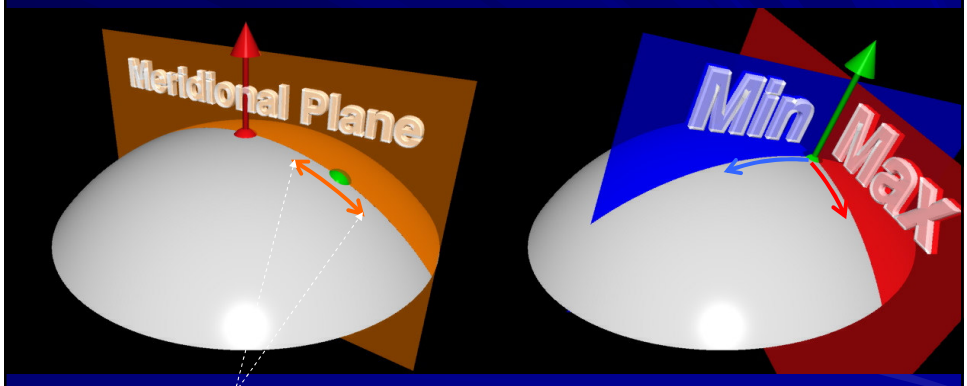
It can be shown that:

$$(\text{Gaussian})^2 = (\text{Mean})^2 + |\text{Astigmatism}|^2$$

Gaussian maps: Cancel astigmatisms …but preserves ectasia



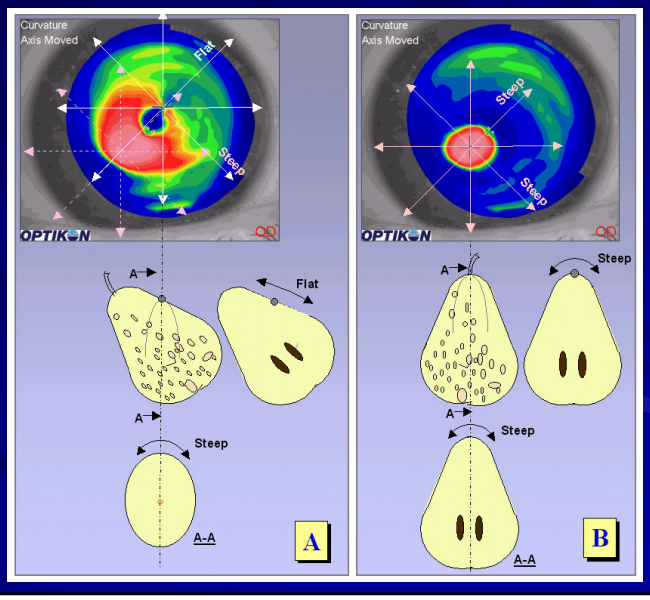
Why “Gaussian” differ from “Curvature” maps?



Instantaneous Curvatures are measured along sections from the VK axis.

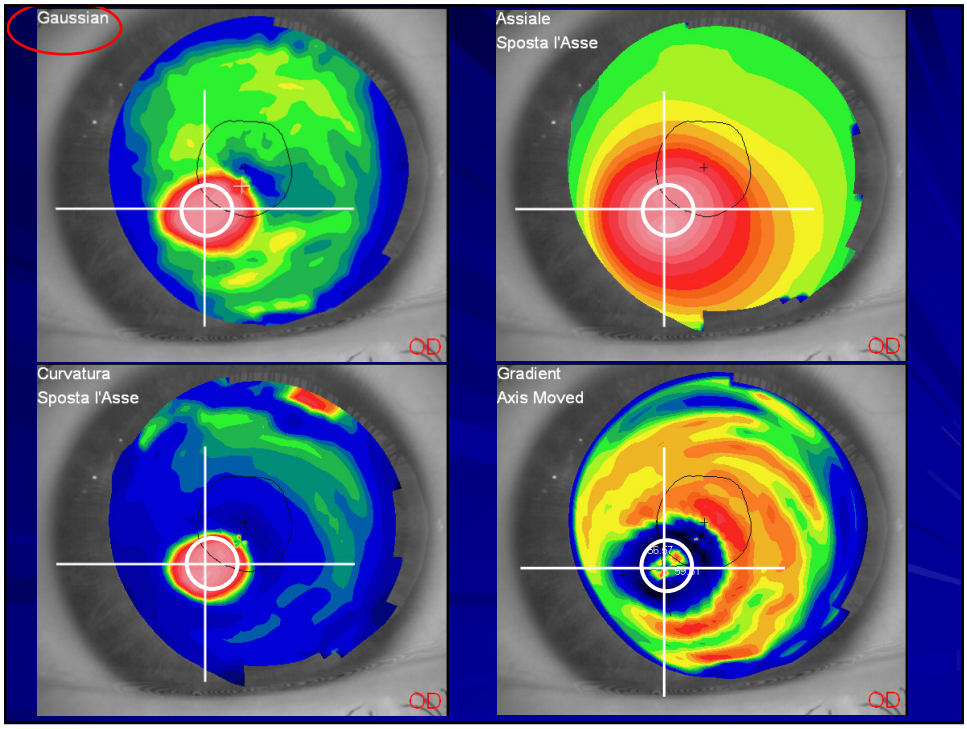
Gaussian are the product of main curvatures at each point and do not depend from VK axis

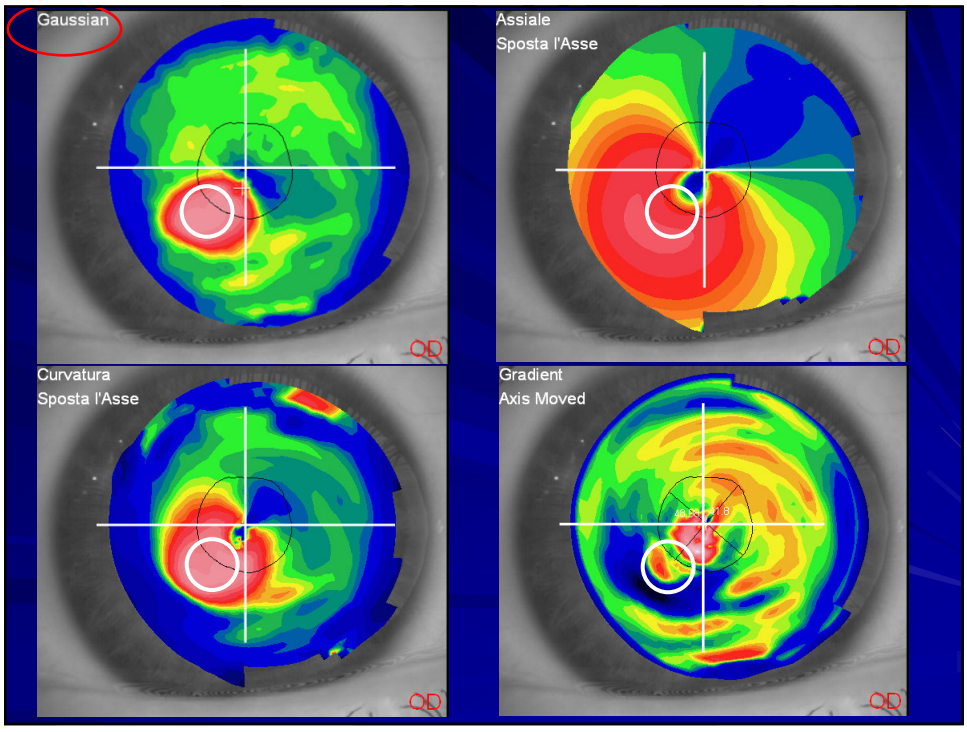
Curvature maps with “Move-Axis”

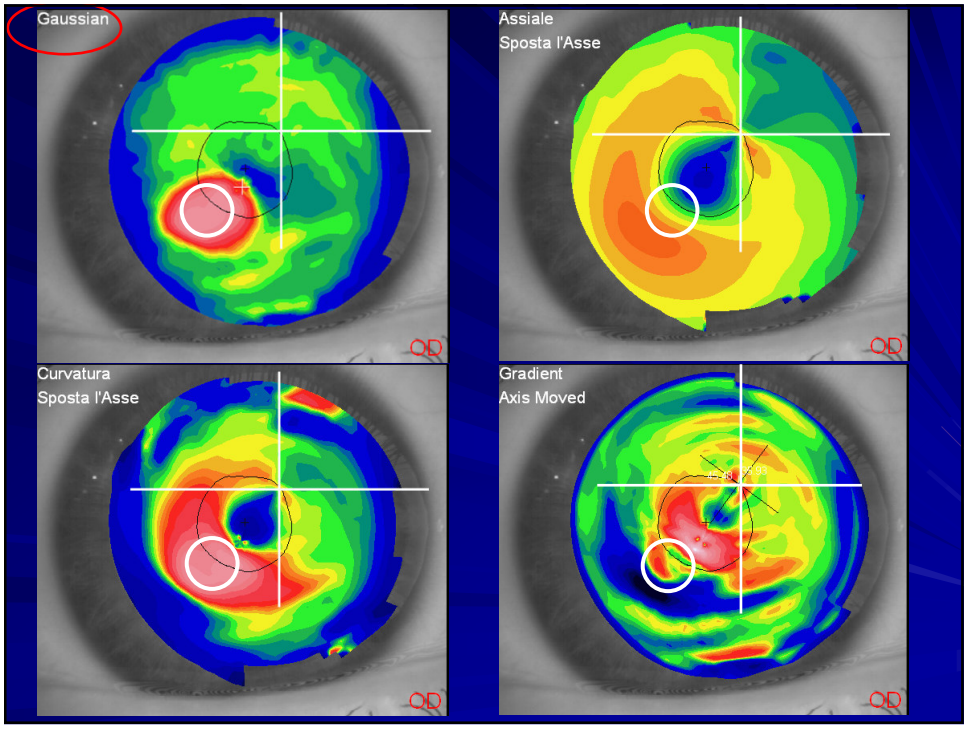


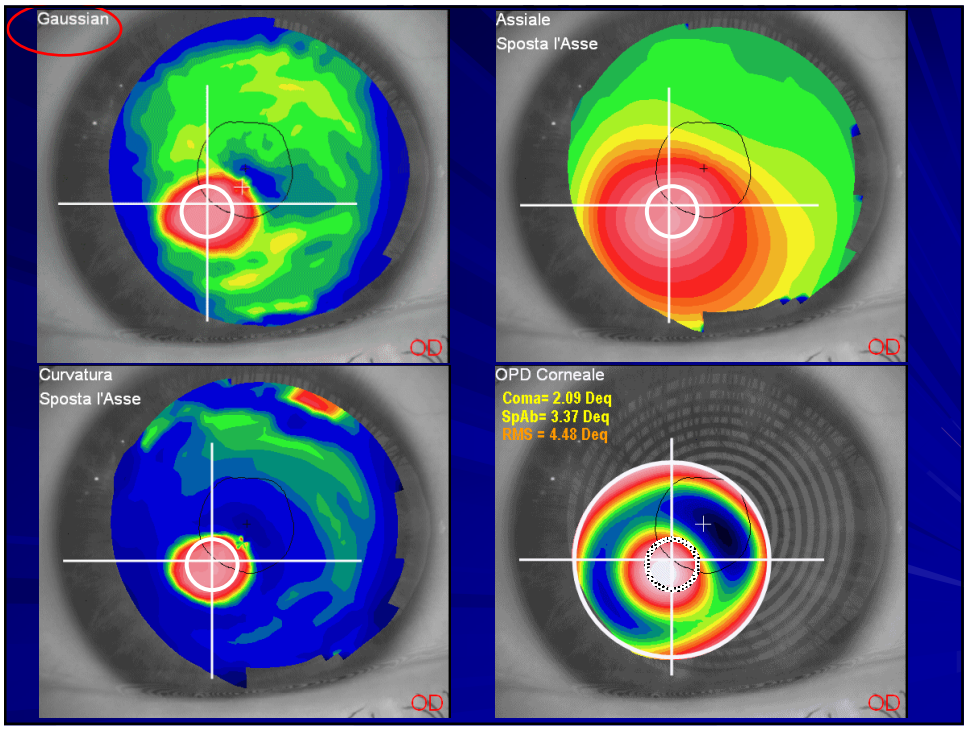
Advantages of Gaussian Maps

- Are indipendent from the VK axis
- Locate corneal apex
- Possible integration with indices (CLMI)


Gaussian
Assiale
Sposta l'Asse
Curvatura
Sposta l'Asse
Gradient
Axis Moved


Gaussian
Assiale
Sposta l'Asse
Curvatura
Sposta l'Asse
Gradient
Axis Moved


Gaussian
Assiale
Sposta l'Asse
Curvatura
Sposta l'Asse
Gradient
Axis Moved
OD

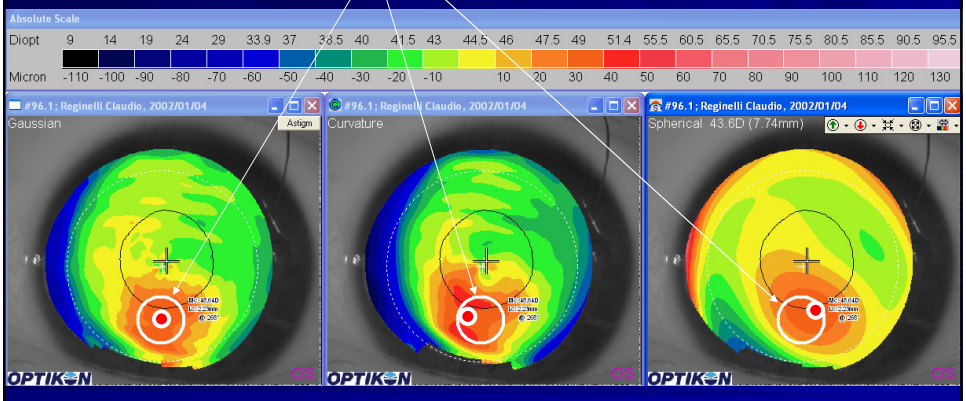

Gaussian
Assiale
Sposta l'Asse
Curvatura
Sposta l'Asse
OPD Corneale
Coma= 2.09 Deq
SpAb= 3.37 Deq
RMS = 4.48 Deq
OD

Advantages of Gaussian Maps

- Are indipendent from the VK axis
- Locate corneal apex (objectively)
- Possible integration with indices (CLMI)

Cone location, with 3 different maps

CLMI calculated on **Gaussian**

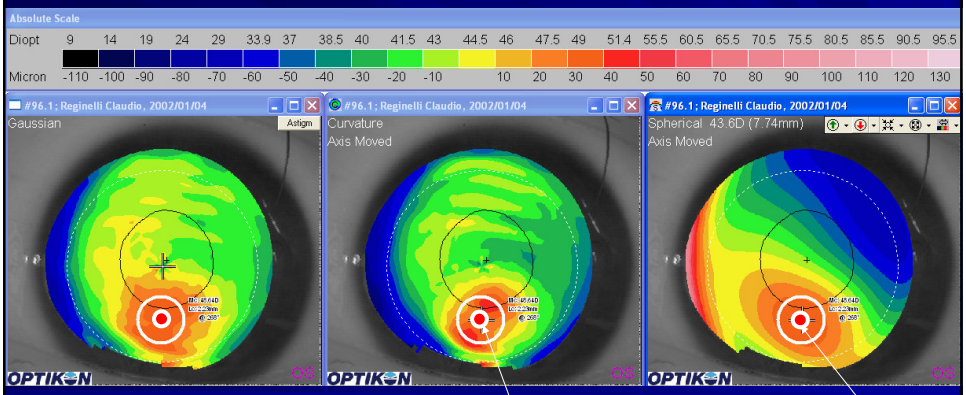


Gaussian map

(Inst.) Curvature map

Sherical Offset **(Best-Fit sphere)**

…same maps, **VK-axis-moved** to the Gaussian CLMI apex



Gaussian map

Curvature (+ Move Axis)

Sherical Offset ("Tangent to point")

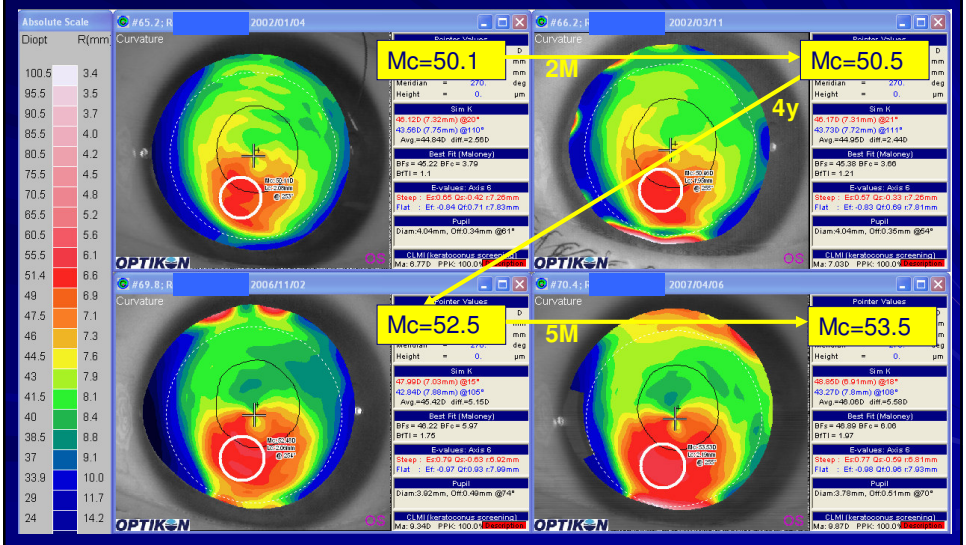
Advantages of Gaussian Maps

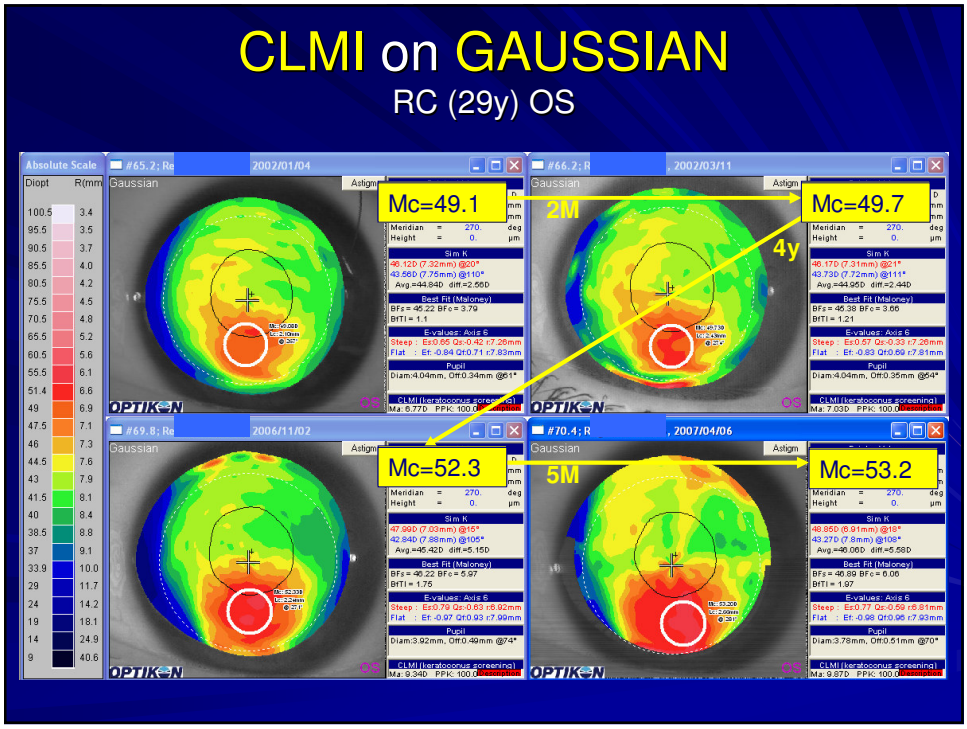
- Are indipendent from the VK axis
- Locate corneal apex objectively
- Possible integration with indices (CLMI)

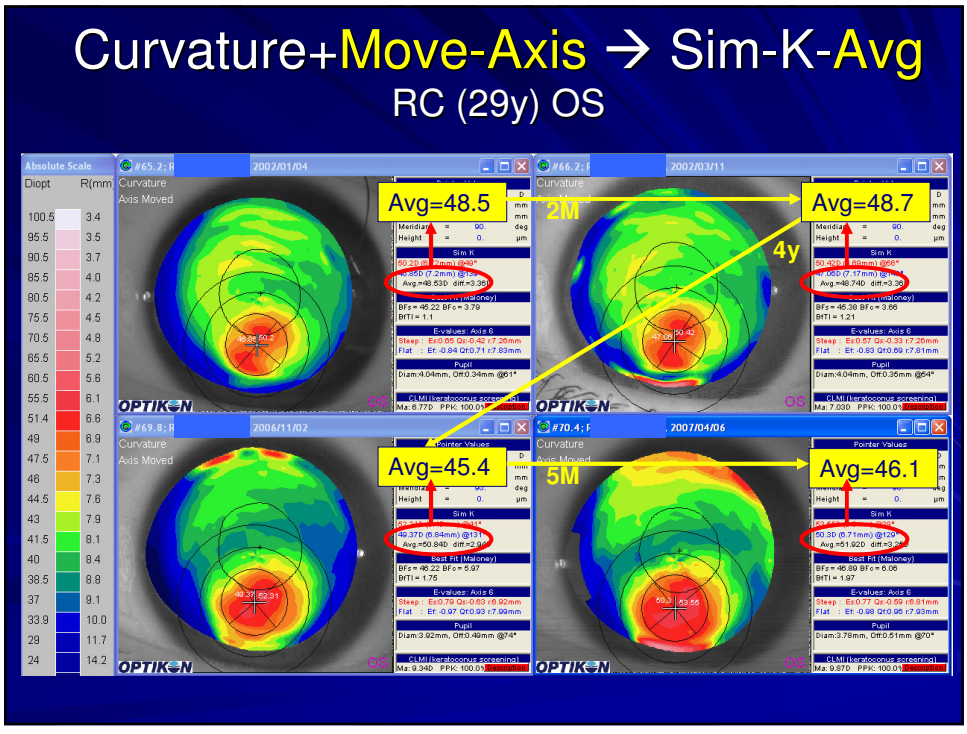
Follow-up of a keratoconus after 5 years

CLMI on Local CURVATURE

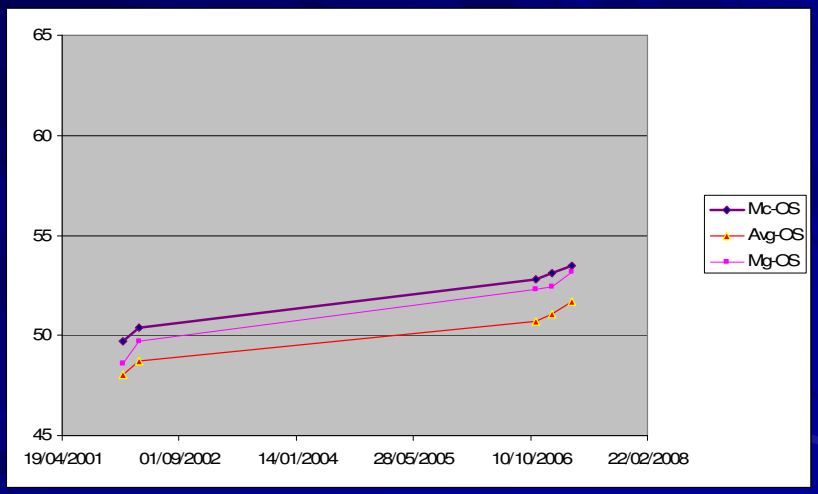
RC (29y) OS




CLMI on GAUSSIAN
RC (29y) OS
Mc=49.1
2M
Mc=49.7
4y
Mc=52.3
5M
Mc=53.2


Curvatura+Move-Axis → Sim-K-Avg
RC (29y) OS
Avg=48.5
2M
Avg=48.7
4y
Avg=45.4
5M
Avg=46.1

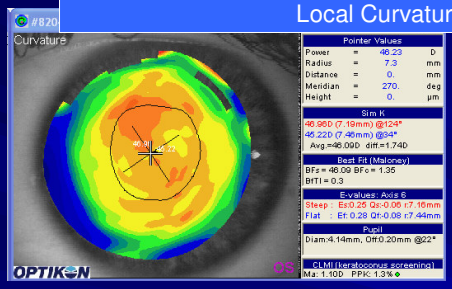
Follow up of RC (29y) OD/OS



Other clinical applications

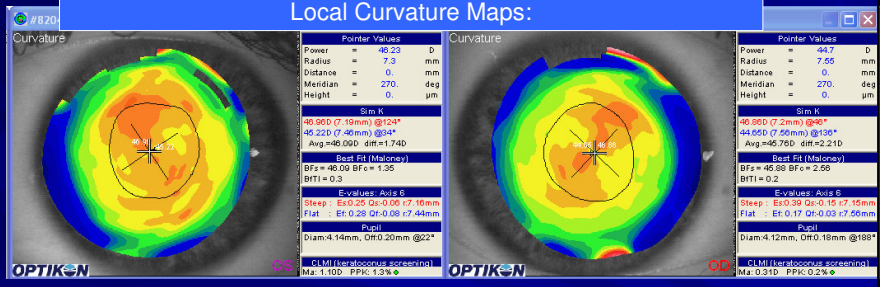
Irregular astigmatism or keratoconus?

Local Curvature Map:

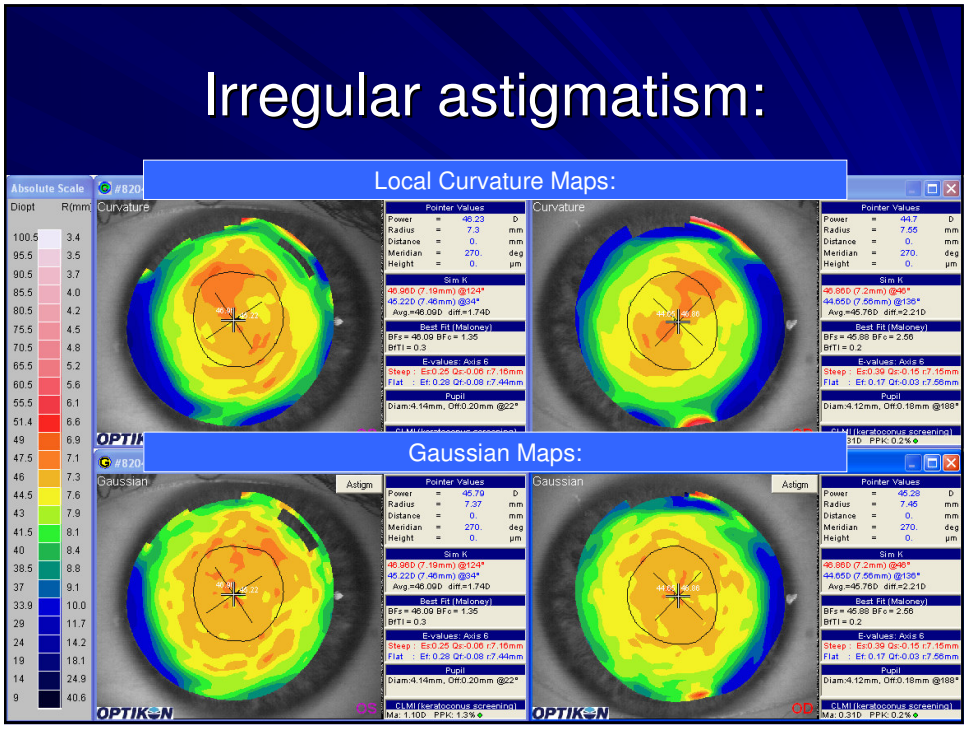


Irregular astigmatism:

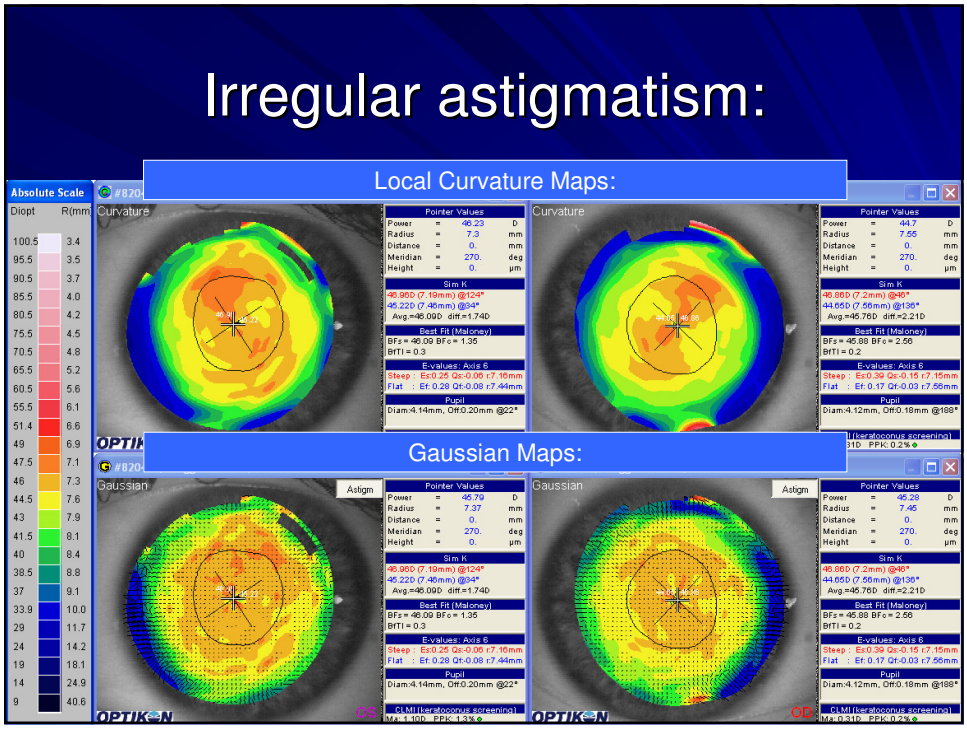
Local Curvature Maps:



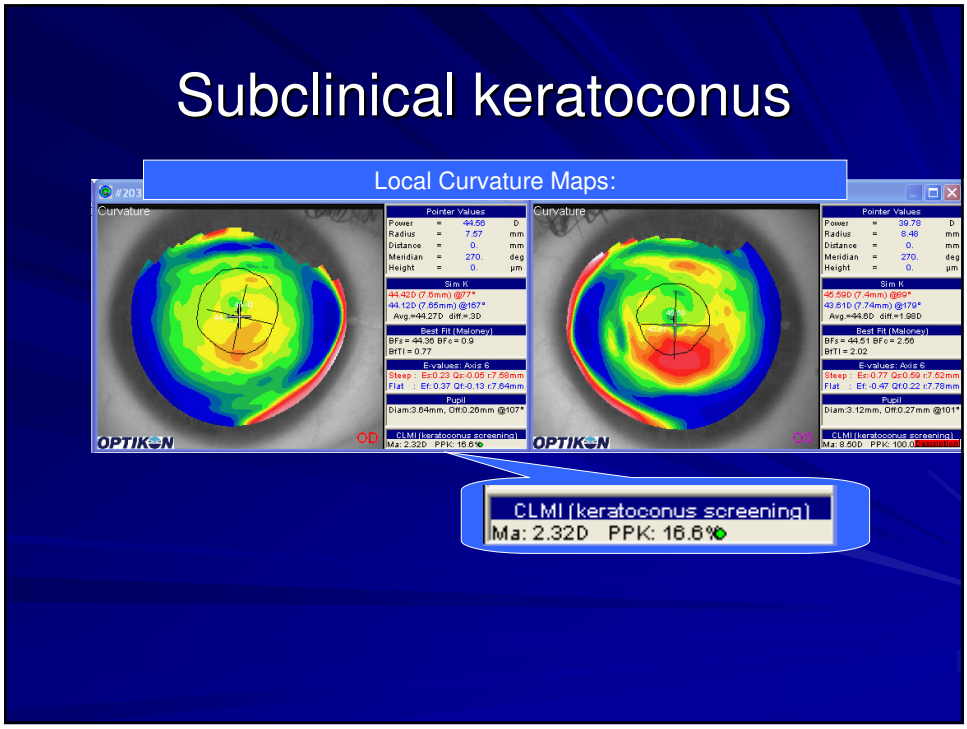
Irregular astigmatism:



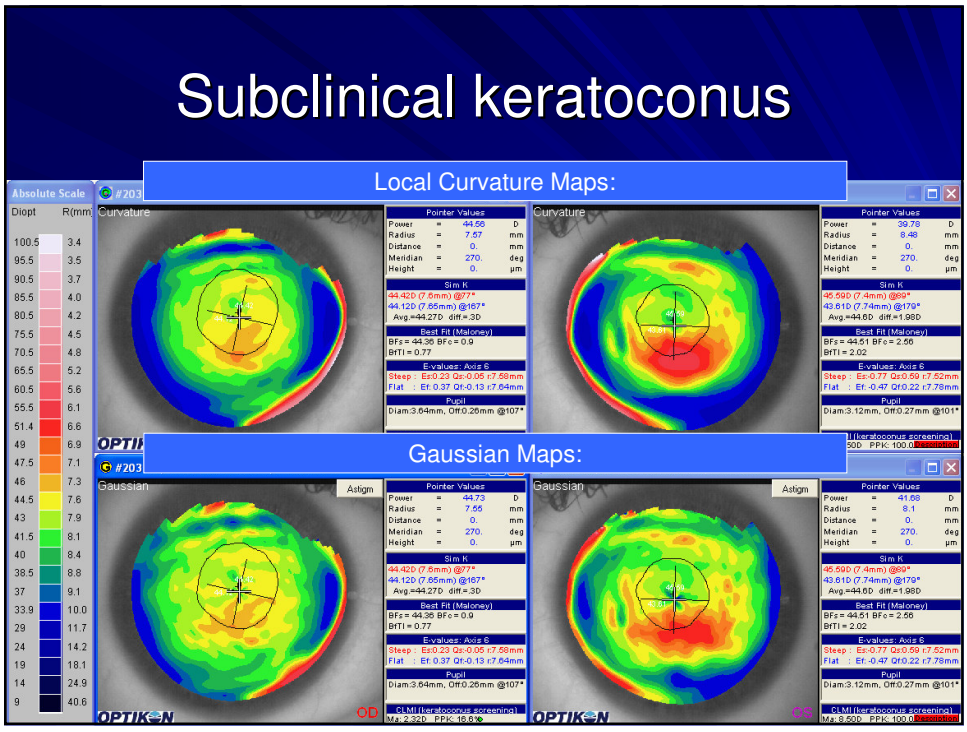
Irregular astigmatism:



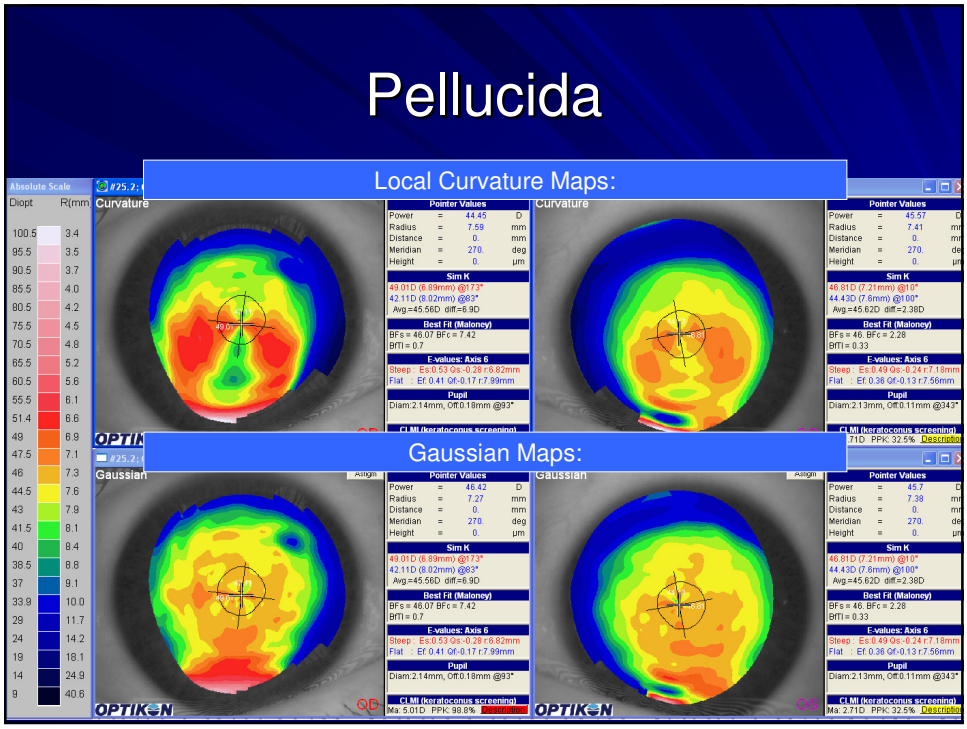
Subclinical keratoconus



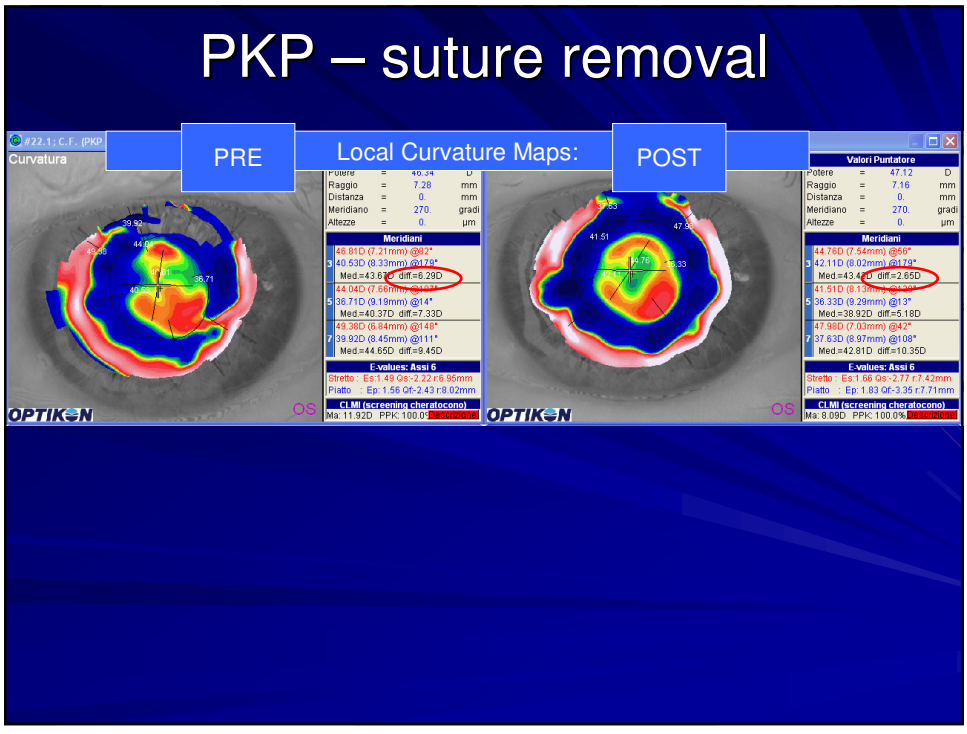
Subclinical keratoconus



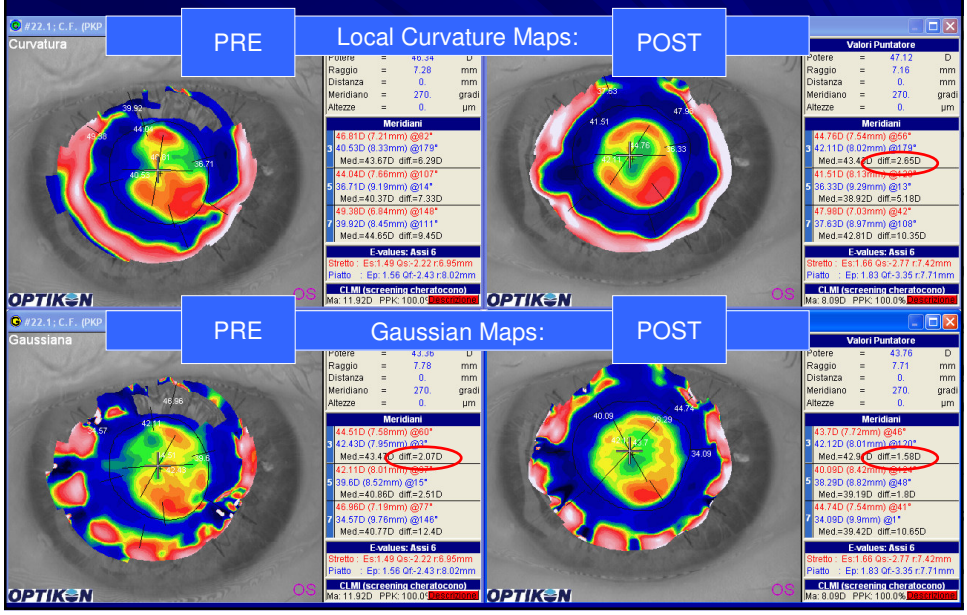
Pellucida



PKP – suture removal



PKP – suture removal



Conclusions, Gaussian Curvatures:

- Represent an intrinsic property of surfaces
- Are “Axis independent”
- Potentially useful in clinical evaluations, and along with CLMI indices
- CAUTION: Do NOT represent shape univoquely (hide astigmatism, show ectasia)
- Recommended together with Instantaneous Curvature and “Move-Axis”

THANK YOU !