The Gaussian Curvature Map

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Historical background:

Theorema Egregium by Carl Friderich Gauss
Historical background:

- C.F. Gauss (1777-1855) was professor of mathematics in Göttingen (D). Has developed theories adopted in:
  - Analysis
  - Number theory
  - Differential Geometry
  - Statistic (Gaussian distribution)
  - Physics (magnetism, electrostatics, optics, Astronomy)

"Almost everything, which the mathematics of our century has brought forth in the way of original scientific ideas, attaches to the name of Gauss."

Kronecker, L.

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Gaussian Curvature of a surface
(Definition)

- Gaussian curvature is the product of steepest and flattest local curvatures, at each point.

| Negative GC | Zero GC | Positive GC |
Gauss’ *Theorema Egregium* (1828)

- Gaussian curvature \((C_1 \times C_2)\) of a flexible surface is invariant. (non-elastic isometric distortion)

\[
C_1 \times C_2 = 0 \quad C_1 \times C_2 = 0
\]

Gauss’ *Theorema Egregium*

- Gaussian curvature is an intrinsic property of surfaces
Gauss’ *Theorema Egregium*

• “A consequence of the Theorema Egregium is that the Earth cannot be displayed on a map without distortion. The Mercator projection... preserves angles but fails to preserve area.” (Wikipedia)

The Gaussian Map in Corneal Topography
Proposed by B. Barsky et al. in 1997

"Gaussian power with cylinder vector field representation for corneal topography maps". *Optom. Vis. Sci.* 74-1997

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**Other proposals**

- T. Turner (Orbscan, 1995) – Mean curvature maps
- R. Mattioli (Keratron, Refractive-on-line 2007) - Gaussian maps

**Mean** is local curvature arithmetic average (D).

**Gaussian** is curvature geometric average (D):

\[
\text{Mean} = \frac{(C_1 + C_2)}{2} \quad \text{Gaussian} = \sqrt{C_1 \times C_2}
\]

It can be shown that:

\[
(Gaussian)^2 = (Mean)^2 + |\text{Astigmatism}|^2
\]
Gaussian maps: **Cancel astigmatisms** …but preserves ectasia

Why “Gaussian” differ from “Curvature” maps?

Instantaneous Curvatures are measured along sections **from the VK axis**.

Gaussian are the product of main curvatures at each point and **do not depend from VK axis**.
Curvature maps with “Move-Axis”

Advantages of Gaussian Maps

- Are independent from the VK axis
- Locate corneal apex
- Possible integration with indices (CLMI)
The Gaussian Curvature
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Advantages of Gaussian Maps

- Are independent from the VK axis
- Locate corneal apex (objectively)
- Possible integration with indices (CLMI)

Cone location, with 3 different maps

CLMI calculated on Gaussian

Gaussian map

(Inst.) Curvature map

Spherical Offset (Best-Fit sphere)
...same maps, **VK-axis-moved** to the Gaussian CLMI apex

Gaussian map

Curvature (+ Move Axis*)

Spherical Offset (*Tangent to point*)

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**Advantages of Gaussian Maps**

- Are independent from the VK axis
- Locate corneal apex objectively
- Possible integration with indices (CLMI)
Follow-up of a keratoconus after 5 years
The Gaussian Curvature

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CLMI on GAUSSIAN
RC (29y) OS

Curvature+Move-Axis → Sim-K-Avg
RC (29y) OS
Follow up of RC (29y) OD/OS

Other clinical applications
Irregular astigmatism or keratoconus?

Irregular astigmatism:

Local Curvature Map:

Local Curvature Maps:
Irregular astigmatism:

Local Curvature Maps:

Gaussian Maps:
Subclinical keratoconus

Local Curvature Maps:

Gaussian Maps:

CLM (keratoconus screening) IMa: 2.32D  PPK: 16.6%
Pellucida

Local Curvature Maps:

Gaussian Maps:

PKP – suture removal

PRE

Local Curvature Maps:

POST
Conclusions, **Gaussian Curvatures:**

- Represent an intrinsic property of surfaces
- Are “Axis independent”
- Potentially useful in clinical evaluations, and along with CLMI indices
- **CAUTION:** Do NOT represent shape univoquely (hide astigmatism, show ectasia)
- Recommended together with Instantaneous Curvature and “Move-Axis”

**THANK YOU!**